#1

For the two level problem, we showed that the Hamiltonian for the undisturbed atom (H_o) is modified by a term describing its interaction with the light field. $H' = -eEr$. The time dependence in H' can be eliminated by making a transformation to the expansion coefficients:

$$c'_g(t) = c_g(t)$$
$$c'_e(t) = c_e(t)e^{-i\delta t}$$

which leads to a transformed time independent H':

$$H' = \frac{\hbar}{2}\begin{bmatrix} -2\delta & \Omega \\ \Omega & 0 \end{bmatrix}$$

Calculate the energy shifts due to H' (called "light shifts") for the ground and excited states. Draw an energy level diagram showing the energies of the states in the absence of the light field, and then showing the shifts due to the light field. Calculate the light shift of the ground state in the limit of $\delta \ll \Omega$ and $\delta \gg \Omega$.

a) Finding the eigenvalues of H'

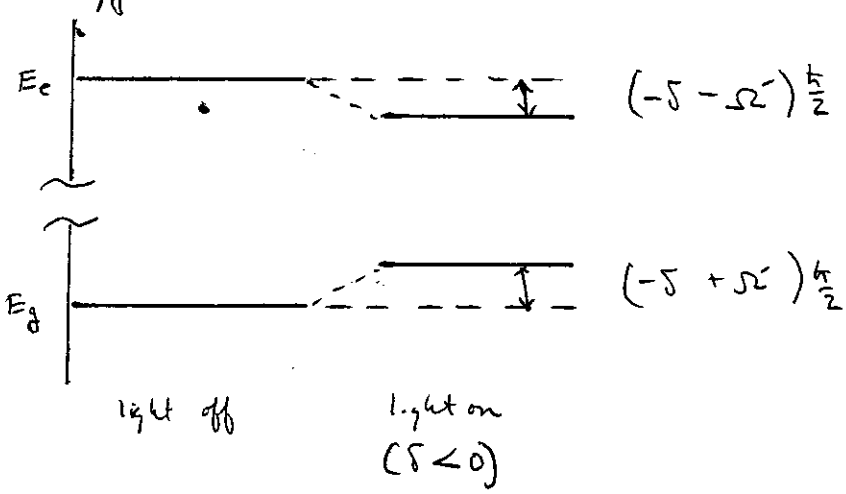
$$\begin{vmatrix} -\hbar\delta - \Delta E & \frac{\hbar\Omega}{2} \\ \frac{\hbar\Omega}{2} & -\Delta E \end{vmatrix} = 0$$

$$\Delta E(\hbar\delta + \Delta E) - \frac{\hbar^2\Omega^2}{4} = 0$$

$$\Delta E = \frac{\hbar}{2}(-\delta \mp \Omega') \quad \text{where } \Omega' = \sqrt{\Omega^2 + \delta^2}$$

Note: the minus sign corresponds to the excited state light shift; the positive sign to the ground state.

b) Energy levels:



#1 cont'd

c) Limits : $|\delta| << \Omega$

We use the binomial theorem:

$$\Delta E = \frac{\hbar}{2}\left(-\delta + \sqrt{\Omega^2+\delta^2}\right) = \frac{\hbar}{2}\left(-\delta + \Omega\sqrt{1+\left(\frac{\delta}{\Omega}\right)^2}\right)$$

$$\simeq \frac{\hbar}{2}\left(\cancel{-\delta}^{\,0} + \Omega\left(1 + \cancel{\frac{1}{2}\frac{\delta^2}{\Omega^2}}^{\,0} + \cdots\right)\right)$$

$$\boxed{\Delta E_g = \frac{\hbar\Omega}{2}} \quad \text{for} \quad |\delta| << \Omega$$

$|\delta| >> \Omega$

$$\Delta E = \frac{\hbar}{2}\left(-\delta + \delta\sqrt{1+\frac{\Omega^2}{\delta^2}}\right)$$

$$\simeq \frac{\hbar}{2}\left(\cancel{-\delta} + \delta\left(\cancel{1} + \frac{1}{2}\frac{\Omega^2}{\delta^2}\right)\right)$$

$$\boxed{\Delta E_g \simeq \frac{\hbar\Omega^2}{4\delta}} \quad \text{for} \quad |\delta| >> \Omega$$

<u>Relevance</u>: To understand the origin of the light shift energy term, and to derive the useful result for sub-doppler cooling, that $\Delta E_g \simeq \frac{\hbar\Omega^2}{4\delta}$ in the low intensity limit. Also to see that in the limit of small detuning, the light shift is independent of the detuning but only depends on the field intensity.

#2

In laser cooling, the polarization of light plays an important role. Different configurations of polarization result in different cooling mechanisms. Suppose two that are linearly polarized with $\vec{E}_1(z)$ perpendicular to $\vec{E}_2(z)$ (e.g., one beam has $\vec{x}$ polarization and the other $\vec{y}$) are counter-propagating along the z-axis.. Write an expression for the total electric field $\vec{E}_{total}(z)$. Describe its spatial variation, and calculate the intensity as a function of z. What do we call this polarization scheme?

a) Perpendicular polarizations : Lin ⊥ Lin (π_x vs. π_y)

$$E = E_0 \hat{x} \cos(\omega_\ell t - kz) + E_0 \hat{y} \cos(\omega_\ell t + kz)$$

$$= E_0 \left[(\hat{x}+\hat{y}) \cos\omega_\ell t \cos kz + (\hat{x}-\hat{y}) \sin\omega_\ell t \sin kz \right]$$

b) Look @ this for select z:

$z = 0$, $E = E_0(\hat{x}+\hat{y}) \cos\omega_\ell t \Rightarrow \pi_{+45°}$ (polarized in the +45° dir)

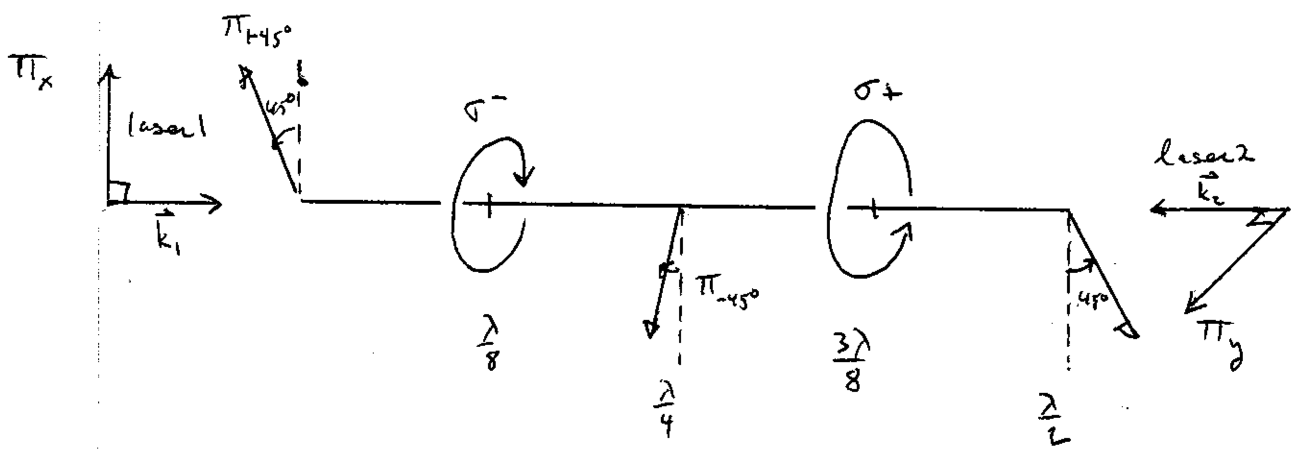
$= \frac{\lambda}{4}$ $kz = \frac{\pi}{2} \Rightarrow E = E_0 (\hat{x}-\hat{y}) \sin\omega_\ell t \Rightarrow \pi_{-45°}$

$= \frac{\lambda}{8}$ $kz = \frac{\pi}{4} \Rightarrow E = E_0 \left[\hat{x} \sin(\omega_\ell t + \frac{\pi}{4}) + \hat{y} \cos(\omega_\ell t + \frac{\pi}{4}) \right]$

x & y components are exactly out of phase, so this represents circularly polarized light rotating about z w/ a negative sense ($\frac{d\varphi}{dt} < 0$) : σ_-

$= \frac{3\lambda}{8}$ $kz = \frac{3\pi}{4}$, σ_+ polarization

Picture:



#2 cont'd

$$\text{Intensity} \equiv \left\langle \frac{|E^2|}{c\mu_0} \right\rangle_{\text{time-averaged}}$$ (From Halliday-Resnick & Walker)

So, $$I(z) = \frac{E_0^2}{c\mu_0} \left| \left\langle \hat{x}\cos^2(\omega_\ell t - kz) + \hat{y}\cos^2(\omega_\ell t + kz) \right\rangle \right|$$

$$= \frac{E_0^2}{c\mu_0} \left| \left\langle \hat{x}\left(\cos\omega_\ell t \cos kz + \sin\omega_\ell t \sin kz\right)^2 + \hat{y}\left(\cos\omega_\ell t\cos kz - \sin\omega_\ell t \sin kz\right)^2 \right\rangle \right|$$

$$\langle \text{squared terms} \rangle = \left\langle \cos^2\omega_\ell t \cos^2 kz + \sin^2\omega_\ell t \sin^2 kz \pm 2\cos\omega_\ell t \cos kz \sin\omega_\ell t \sin kz \right\rangle$$

$$= \cos^2 kz \underbrace{\int_0^{\frac{2\pi}{\omega_\ell}} \cos^2\omega_\ell t}_{=\frac{1}{2}} + \sin^2 kz \underbrace{\int_0^{2\pi/\omega_\ell} \sin^2\omega_\ell t}_{=\frac{1}{2}} \pm 2\cos kz \sin kz \underbrace{\int_0^{\frac{2\pi}{\omega_\ell}} \cos\omega_\ell t \sin\omega_\ell t}_{=\frac{1}{2\omega_\ell}\sin^2\omega_\ell t\Big|_0^{\frac{2\pi}{\omega}} = 0}$$

$$= \frac{\cos^2 kz}{2} + \frac{\sin^2 kz}{2} = \frac{1}{2}$$

So, $$I(z) = \frac{E_0^2}{c\mu_0}\left(\frac{1}{2} + \frac{1}{2}\right) = \boxed{\frac{E_0^2}{c\mu_0}}$$

→ Time averages to a constant Intensity, independent of z.

<u>Relevance</u>: Develop insight into the behavior of different polarization schemes used to achieve sub-doppler cooling. We also see that the intensity of the standing wave has no "nodes"; it is spatially constant. Thus the light shift for atomic states in resonance w/ this light spatially varies only because of the polarization coupling, not the intensity of the field.

#3

Now suppose the two counter-propagating beams have opposite circular polarizations. Write an expression for the electric field $\vec{E}(z)$. Show that the intensity, proportional to $|\vec{E}(z)|^2$, is independent of z. Describe the behavior of the field as a function of z. What do we call this polarization scheme?

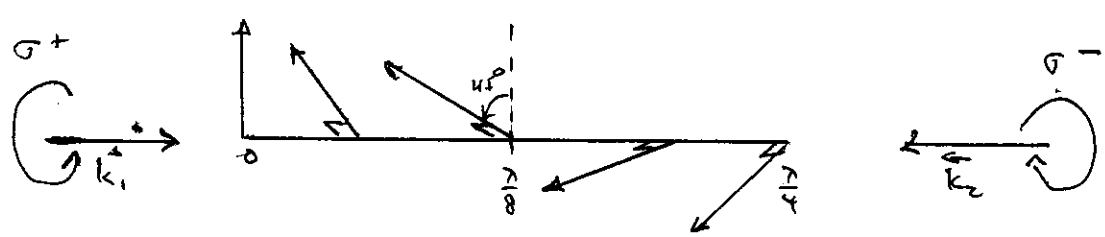
a)
$$\vec{E} = E_0\left[\hat{x}\cos(\omega_e t - kz) + \hat{y}\sin(\omega_e t - kz)\right] + \quad (\leftarrow \sigma_+ \text{ from } -z)$$
$$E_0\left[\hat{x}\cos(\omega_e t + kz) - \hat{y}\sin(\omega_e t + kz)\right] \quad (\leftarrow \sigma_- \text{ from } +z)$$
$$= E_0\left[\hat{x}\left(\cos(\omega_e t - kz) + \cos(\omega_e t + kz)\right) + \hat{y}\left(\sin(\omega_e t - kz) - \sin(\omega_e t + kz)\right)\right]$$

$\hat{x}$:
$$= \cos\omega_e t\cos kz + \cancel{\sin\omega_e t\sin kz} + \cos\omega_e t\cos kz - \cancel{\sin\omega_e t\sin kz}$$
$$= 2\cos\omega_e t\cos kz$$

$\hat{y}$:
$$= \cancel{\sin\omega_e t\cos kz} + \cos\omega_e t\sin kz - \cancel{\sin\omega_e t\cos kz} + \cos\omega_e t\sin kz$$
$$= 2\cos\omega_e t\sin kz$$

So,
$$\vec{E} = 2E_0\cos\omega_e t\left[\hat{x}\cos kz + \hat{y}\sin kz\right]$$

This represents a linearly polarized field whose direction rotates about the z-axis as a function of z. The magnitude is constant:



$$I(z) = \frac{4E_0^2}{c\mu_0}\underbrace{\langle\cos^2\omega_e t\rangle}_{=\frac{1}{2}}\left[\underbrace{\cos^2 kz + \sin^2 kz}_{=1}\right] = \boxed{\frac{2E_0^2}{c\mu_0}}$$

<u>Relevance</u>: See #2.

#4

***Problem 11.1 **Rutherford scattering.** An incident particle of charge q_1 and kinetic energy E scatters off a heavy stationary particle of charge q_2.

~~(a)~~ Derive the formula relating the impact parameter to the scattering angle.[2] *Answer:* $b = (q_1 q_2 / 8\pi\epsilon_0 E)\cot(\theta/2)$.

(b) Determine the differential scattering cross-section. *Answer:*

$$D(\theta) = \left[\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)}\right]^2. \qquad [11.11]$$

(c) Show that the total cross-section for Rutherford scattering is *infinite*. We say that the $1/r$ potential has "infinite range"; you can't escape from a Coulomb force.

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b) $D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$. We will use b from part (a)

$$\frac{db}{d\theta} = \frac{q_1 q_2}{8\pi\varepsilon_0 E}\left(-\frac{1}{2\sin^2\left(\frac{\theta}{2}\right)}\right) \quad \text{\&} \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

So,

$$D(\theta) = \frac{1}{2\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}\left(\frac{q_1 q_2}{8\pi\varepsilon_0 E}\,\frac{\cancel{\cos\theta/2}}{\sin\theta/2}\right)\cdot\frac{q_1 q_2}{8\pi\varepsilon_0 E}\,\frac{1}{2\sin^2\theta/2}$$

$$= \boxed{\frac{q_1 q_2}{16\pi\varepsilon_0 E\sin^2\frac{\theta}{2}}}$$

c) $$\sigma = \int D(\theta)\sin\theta\, d\theta\, d\varphi = 2\pi\left(\frac{q_1 q_2}{16\pi\varepsilon_0 E}\right)^2\int_0^\pi \frac{\sin\theta}{\sin^4\left(\frac{\theta}{2}\right)}\,d\theta$$

This integral does not converge, for near $\theta = 0$ (and again near π), we have $\sin\theta \approx \theta$ & $\sin\frac{\theta}{2} \approx \frac{\theta}{2}$

So, $$16\int_0^\epsilon \theta^{-3}\,d\theta = -8\,\theta^{-2}\Big|_0^\epsilon \longrightarrow \infty$$

<u>Relevance</u>: To gain facility w/ the cross section concept & remind ourselves of the nature of Rutherford scattering.

We note that the <u>total</u> cross section for Rutherford scattering is infinite $\Rightarrow$ "you can't escape from a Coulomb force."