1

Problem 6.16 Evaluate the following commutators: (a) $[\mathbf{L}\cdot\mathbf{S},\mathbf{L}]$, (b) $[\mathbf{L}\cdot\mathbf{S},\mathbf{S}]$, (c) $[\mathbf{L}\cdot\mathbf{S},\mathbf{J}]$, (d) $[\mathbf{L}\cdot\mathbf{S},L^2]$, (e) $[\mathbf{L}\cdot\mathbf{S},S^2]$, (f) $[\mathbf{L}\cdot\mathbf{S},J^2]$. *Hint:* $\mathbf{L}$ and $\mathbf{S}$ satisfy the fundamental commutation relations for angular momentum (Equations 4.99 and 4.134), but they commute with each other.

(a) $[L\cdot S, L]$: Need to determine 3 commutators, for L_x, L_y & L_z:

$[L\cdot S, L_x] = [L_xS_x + L_yS_y + L_zS_z, L_x]$

Since S & L operate in different spaces, they commute automatically. So,

$= S_x[L_x,L_x] + S_y[L_y,L_x] + S_z[L_z,L_x]$

$= S_x(0) + S_y(-i\hbar L_z) + S_z(i\hbar L_y)$

$= i\hbar(L_yS_z - L_zS_y) = i\hbar(L\times S)_x$

Same goes for the other two components, so

$\boxed{[L\cdot S, L] = i\hbar(L\times S)}$

(b) $[L\cdot S, S]$ is identical, only w/ $L \longleftrightarrow S$:

$\boxed{[L\cdot S, S] = i\hbar(S\times L)}$

(c) $[L\cdot S, J] = [L\cdot S, L] + [L\cdot S, S]$

$= i\hbar(L\times S + S\times L) = \boxed{0}$

(d) $[L\cdot S, L^2]$ L^2 commutes w/ L_x, L_y & L_z, (and S) so

$\boxed{[L\cdot S, L^2] = 0}$

(e) Likewise $\boxed{[L\cdot S, S^2] = 0}$

(f) $[L\cdot S, J^2] = [L\cdot S, L^2] + [L\cdot S, S^2] = \boxed{0}$

* Relevance: In deriving the Energy splitting for Spin-orbit coupling, it was pointed out that spin & orbital angular momentum were no longer seperately conserved, as the spin-orbit perturbation Hamiltonian, $H_{S.O}'$ does not commute w/ $\vec{L}$ & $\vec{S}$.

#1 cont'd

Since H'_{SO} depends on the operator $\vec{S}\cdot\vec{L}$, this boils down to showing that $S\cdot L$ does not commute w/ $\vec{L}$ or $\vec{S}$. However, $\vec{S}\cdot\vec{L}$ does commute w/ L^2, S^2 & J^2, so eigenstates of these operators are good eigenstates for evaluating the perturbation. In this problem, you are asked to prove the asserted commutation relations state here.

#2

***Problem 6.17** Derive the fine structure formula (Equation 6.66) from the relativistic correction (Equation 6.57) and the spin-orbit coupling (Equation 6.65). *Hint:* Note that $j = l \pm 1/2$; treat the plus sign and the minus sign separately, and you'll find that you get the same final answer either way.

With the plus sign, $j = l + \frac{1}{2} \rightarrow l = j - \frac{1}{2}$

So Eq. 6.57 becomes:

$$E_r' = -\frac{(E_n)^2}{2mc^2}\left(\frac{4n}{j} - 3\right)$$

Eq. 6.65 becomes:

$$E_{so}' = \frac{(E_n)^2}{mc^2}\frac{n\left[j(j+1) - (j-\frac{1}{2})(j+\frac{1}{2}) - \frac{3}{4}\right]}{(j-\frac{1}{2})(j)(j+\frac{1}{2})}$$

$$= \frac{(E_n)^2}{mc^2}\frac{n\left(j^2 + j - j^2 + \frac{1}{4} - \frac{3}{4}\right)}{(j-\frac{1}{2})j(j+\frac{1}{2})}$$

$$= \frac{(E_n)^2}{mc^2}\frac{n\,\cancel{(j-\frac{1}{2})}}{\cancel{(j-\frac{1}{2})}\,j(j+\frac{1}{2})} = \frac{(E_n)^2}{mc^2}\frac{n}{j(j+\frac{1}{2})}$$

$$\therefore\quad E_{fs}' = E_r' + E_{so}' = \frac{(E_n)^2}{2mc^2}\left(-\frac{4n}{j} + 3 + \frac{2n}{j(j+\frac{1}{2})}\right)$$

$$= \frac{(E_n)^2}{2mc^2}\left(3 + \frac{\cancel{2n} - 4n(j+\cancel{\tfrac{1}{2}})}{j(j+\frac{1}{2})}\right)$$

$$= \boxed{\frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right)} \quad ✓$$

Now with the minus sign: $j = l - \frac{1}{2} \rightarrow l = j + \frac{1}{2}$

Eq. 6.57:

$$E_r' = -\frac{(E_n)^2}{2mc^2}\left(\frac{4n}{j+1} - 3\right)$$

Eq 6.65:

$$E_{so}' = \frac{(E_n)^2}{mc^2}\frac{n\left[j(j+1) - (j+\frac{1}{2})(j+\frac{3}{2}) - \frac{3}{4}\right]}{(j+\frac{1}{2})(j+1)(j+\frac{3}{2})}$$

#2 cont'd

$$E'_{so} = \frac{(E_n)^2}{mc^2} \frac{n\left(j^2 + j - j^2 - 2j - \frac{3}{4} - \frac{3}{4}\right)}{(j+\frac{1}{2})(j+1)(j+3/2)}$$

$$= \frac{(E_n)^2}{mc^2} \frac{(-n)\cancel{(j+3/2)}}{(j+\frac{1}{2})(j+1)\cancel{(j+3/2)}} = \frac{(E_n^2)}{mc^2} \frac{-n}{(j+\frac{1}{2})(j+1)}$$

so

$$E'_{fs} = \frac{(E_n)^2}{2mc^2}\left[-\frac{4n}{j+1} + 3 - \frac{2n}{(j+1)(j+\frac{1}{2})}\right]$$

$$= \frac{(E_n)^2}{2mc^2}\left\{3 - \frac{2n + 4n(j+\frac{1}{2})}{(j+1)(j+\frac{1}{2})}\right\}$$

$$= \frac{(E_n)^2}{2mc^2}\left\{3 - \frac{4n\cancel{(j+1)}}{\cancel{(j+1)}(j+\frac{1}{2})}\right\}$$

$$= \boxed{\frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right)} \checkmark$$

Relevance: This exercise exhibits the fact that the spin-orbit & relativistic corrections are of the same order. Also, the combined answer results in a great simplification, dependent only on n & j, but not l.

#3

***Problem 6.21** Consider the (eight) $n = 2$ states, $|2ljm_j\rangle$. Find the energy of each state, under weak-field Zeeman splitting, and construct a diagram like Figure 6.11 to show how the energies evolve as B_{ext} increases. Label each line clearly, and indicate its slope.

For $n=2$, $l=0$ $(j=\frac{1}{2})$ or $l=1$ $(j=\frac{1}{2}, \frac{3}{2})$ the eight states are:

$$\left.\begin{array}{l} |1\rangle = |2\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |2\rangle = |2\,0\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \end{array}\right\} \quad g_J = \left[1 + \frac{(\frac{1}{2})(\frac{3}{2}) + (\frac{3}{4})}{2(\frac{1}{2})(\frac{3}{2})}\right] = 1 + \frac{3/2}{3/2} = 2$$

$$\left.\begin{array}{l} |3\rangle = |2\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |4\rangle = |2\,1\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \end{array}\right\} \quad g_J = \left[1 + \frac{(\frac{1}{2})(\frac{3}{2}) - (\frac{1}{2}) + (\frac{3}{4})}{2(\frac{1}{2})(\frac{3}{2})}\right] = 1 + \frac{-1/2}{3/2} = \frac{2}{3}$$

In these four cases, $E_{nj} = -\frac{13.6\,\text{eV}}{4}\left[1 + \frac{\alpha^2}{4}\left(\frac{2}{1} - \frac{3}{4}\right)\right] = -3.4\,\text{eV}\left(1 + \frac{5}{16}\alpha^2\right)$.

← Energy levels of H w/ Fine structure

$$\left.\begin{array}{l} |5\rangle = |2\,1\,\tfrac{3}{2}\,\tfrac{3}{2}\rangle \\ |6\rangle = |2\,1\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle \\ |7\rangle = |2\,1\,\tfrac{3}{2}\,-\tfrac{1}{2}\rangle \\ |8\rangle = |2\,1\,\tfrac{3}{2}\,-\tfrac{3}{2}\rangle \end{array}\right\} \quad g_J = \left[1 + \frac{(\frac{3}{2})(\frac{5}{2}) - (1)(2) + (\frac{3}{4})}{2(\frac{3}{2})(\frac{5}{2})}\right] = 1 + \frac{5/2}{15/2} = \frac{4}{3}$$

In these four cases, $E_{nj} = -3.4\,\text{eV}\left[1 + \frac{\alpha^2}{4}\left(\frac{2}{2} - \frac{3}{4}\right)\right] = -3.4\,\text{eV}\left(1 + \frac{1}{16}\alpha^2\right)$

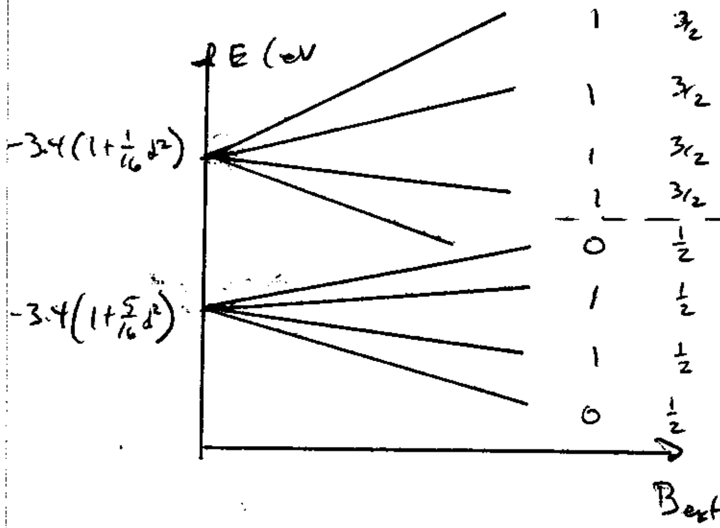
Now we add the Zeeman energy:

$$E_Z' = g_J\, m_j\, \mu_B B_{ext}$$

So the total Energies are:

$$E_1 = -3.4\,\text{eV}\left(1 + \tfrac{5}{16}\alpha^2\right) + \mu_B B_{ext}$$
$$E_2 = -3.4\,\text{eV}\left(1 + \tfrac{5}{16}\alpha^2\right) - \mu_B B_{ext}$$
$$E_3 = -3.4\,\text{eV}\left(1 + \tfrac{5}{16}\alpha^2\right) + \tfrac{1}{3}\mu_B B_{ext}$$
$$E_4 = -3.4\,\text{eV}\left(1 + \tfrac{5}{16}\alpha^2\right) - \tfrac{1}{3}\mu_B B_{ext}$$
$$E_5 = -3.4\,\text{eV}\left(1 + \tfrac{1}{16}\alpha^2\right) + 2\mu_B B_{ext}$$
$$E_6 = -3.4\,\text{eV}\left(1 + \tfrac{1}{16}\alpha^2\right) + \tfrac{2}{3}\mu_B B_{ext}$$
$$E_7 = -3.4\,\text{eV}\left(1 + \tfrac{1}{16}\alpha^2\right) - \tfrac{2}{3}\mu_B B_{ext}$$
$$E_8 = -3.4\,\text{eV}\left(1 + \tfrac{1}{16}\alpha^2\right) - 2\mu_B B_{ext}$$

#3 (cont'd)



l	j	m_j	Slope
1	$3/2$	$3/2$	2
1	$3/2$	$\frac{1}{2}$	$\frac{2}{3}$
1	$3/2$	$-\frac{1}{2}$	$-\frac{2}{3}$
1	$3/2$	$-3/2$	-2
0	$\frac{1}{2}$	$\frac{1}{2}$	1
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$
1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$
0	$\frac{1}{2}$	$-\frac{1}{2}$	-1

Relevance:

This problem gives us experience in the behavior of the Zeemann effect & nature of the degeneracy-breaking that occurs under the application of a magnetic field. We note that the dependence of the energy shift on m_j causes the unperturbed states to split into a symmetric set of energy levels, half being positive energy shifts (corresponding to positive values of m_j) & negative energy shifts (corresponding to negative values of m_j).