#1

*__Problem 6.1__ Suppose we put a delta-function bump in the center of the infinite square well:

$$H' = \alpha\delta(x - a/2),$$

where α is a constant.

(a) Find the first-order correction to the allowed energies. Explain why the energies are not perturbed for even n.

(b) Find the first three nonzero terms in the expansion (Equation 6.13) of the correction to the ground state, ψ_1^1.

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a) The unperturbed wavefunctions are:

$$|\psi_n^0(x)\rangle = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \quad \text{\& } \quad H' = \alpha\delta(x-a/2)$$

So,

$$\begin{aligned} E_n^1 &= \langle\psi_n^0|H'|\psi_n^0\rangle \\ &= \frac{2}{a}\alpha\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)\delta\left(x-\frac{a}{2}\right)dx \\ &= \frac{2\alpha}{a}\sin^2\left(\frac{n\pi}{a}\frac{a}{2}\right) \\ &= \frac{2\alpha}{a}\sin\left(\frac{n\pi}{2}\right) = \boxed{\begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2\alpha}{a} & \text{if } n \text{ is odd}\end{cases}} \end{aligned}$$

For even n, the wave function is zero at the location of the perturbation ($x = \frac{a}{2}$), so it never "feels" H'.

b) Here $n=1$, & we need to evaluate:

$$\psi_1^1 = \sum_{m\neq 1}\frac{\langle\psi_m^0|H'|\psi_1^0\rangle}{(E_1^0 - E_m^0)}\psi_m^0$$

Starting w/ the matrix element $\langle\psi_m^0|H'|\psi_1^0\rangle$:

$$\begin{aligned}\langle\psi_m^0|H'|\psi_1^0\rangle &= \frac{2\alpha}{a}\int \sin\left(\frac{m\pi}{a}x\right)\delta\left(x-\frac{a}{2}\right)\sin\left(\frac{\pi}{a}x\right)dx \\ &= \frac{2\alpha}{a}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = \frac{2\alpha}{a}\sin\left(\frac{m\pi}{2}\right)\end{aligned}$$

This is zero for even m, so the 1st 3 non-zero terms will be $m = 3, 5$ & 7.

#1 b) Cont'd

Meanwhile,

$$E_1^\circ - E_m^\circ = \frac{\pi^2 \hbar^2}{2ma^2}(1-m^2) \quad \text{so,}$$

$$\psi_1' = \sum_{m=3,5,7\ldots} \frac{(2\alpha/a)\sin \pi m/2}{E_1^\circ - E_m^\circ}\psi_m^\circ$$

$$= \frac{2\alpha}{a}\frac{2ma^2}{\pi^2\hbar^2}\left[\frac{-1}{1-9}\psi_3^\circ + \frac{1}{1-25}\psi_5^\circ + \frac{-1}{1-49}\psi_7^\circ + \cdots\right]$$

$$= \frac{4m\alpha a}{\pi^2\hbar^2}\sqrt{\frac{2}{a}}\left[\frac{1}{8}\sin\left(\frac{3\pi x}{a}\right) - \frac{1}{24}\sin\left(\frac{5\pi}{a}x\right) + \frac{1}{48}\sin\left(\frac{7\pi}{a}x\right) + \cdots\right]$$

$$= \boxed{\frac{m\alpha}{\pi^2\hbar^2}\sqrt{\frac{a}{2}}\left[\sin\left(\frac{3\pi x}{a}\right) - \frac{1}{3}\sin\left(\frac{5\pi x}{a}\right) + \frac{1}{6}\sin\left(\frac{7\pi x}{a}\right) + \cdots\right]}$$

This problem is intended to give you experience in applying 1st order time-independent perturbation theory to a specific problem. Although the delta-function perturbation may be unphysical, it does make it easier to compute the solution (by getting rid of the integral), so you can concentrate on the application of PT, rather than calculus.

#2

Problem 6.5 Consider a charged particle in the one-dimensional harmonic oscillator potential. Suppose we turn on a weak electric field (E), so that the potential energy is shifted by an amount $H' = -qEx$.

(a) Show that there is no first-order change in the energy levels, and calculate the second-order correction. *Hint:* See Problem 3.33.

(b) The Schrödinger equation can be solved directly in this case, by a change of variables: $x' \equiv x - (qE/m\omega^2)$. Find the exact energies, and show that they are consistent with the perturbation theory approximation.

a) First order energy correction:

$$E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle = -qE \langle n | x | n \rangle$$

recall that: $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$

$$a_+ |n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$a_- |n\rangle = \sqrt{n}\,|n-1\rangle$$

So, $\langle n | x | n \rangle = \sqrt{\frac{\hbar}{2m\omega}}$

but $\langle n | n' \rangle = \delta_{n,n'}$, so $\langle n | n+1 \rangle = 0$ & $\langle n | n-1 \rangle = 0$

$$\therefore \boxed{E_n^1 = 0}$$

The Second order energy correction is given by:

$$E_n^2 = (qE)^2 \sum_{m \neq n} \frac{|\langle m | x | n \rangle|^2}{(n-m)\hbar\omega} \qquad \left(\text{we used } E_n^0 = (n+\tfrac{1}{2})\hbar\omega\right)$$

From problem 3.33, $\langle m | x | n \rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{m,n-1} + \sqrt{m}\,\delta_{n,m-1}\right)$

So,

$$E_n^2 = \frac{(qE)^2}{\hbar\omega}\,\frac{\hbar}{2m\omega} \sum_{m \neq n} \frac{|\sqrt{n}\,\delta_{m,n-1} + \sqrt{m}\,\delta_{n,m-1}|^2}{(n-m)}$$

$$= \frac{(qE)^2}{2m\omega^2}\left[\underbrace{\frac{(\sqrt{n})^2}{n-(n-1)}}_{m=n-1} + \underbrace{\frac{(\sqrt{n+1})^2}{n-(n+1)}}_{m=n+1}\right]$$

2, a) cont'd:

$$E_n^2 = \frac{(qE)^2}{2m\omega^2}\Big[n - (n+1) \Big] = \boxed{-\frac{(qE)^2}{2m\omega^2}}$$

b) Exact Solution:

Schrodinger Equation: $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left(\frac{1}{2}m\omega^2x^2 - qEx\right)\psi = E\psi$$

Let $x' = x - \frac{qE}{m\omega^2}$. Then,

$$\frac{1}{2}m\omega^2x^2 - qEx = \frac{1}{2}m\omega^2\left[x' + \left(\frac{qE}{m\omega^2}\right)\right]^2 - qE\left[x' + \left(\frac{qE}{m\omega^2}\right)\right]$$

$$= \frac{1}{2}m\omega^2x'^2 + \cancel{m\omega^2 x' \frac{qE}{m\omega^2}} + \frac{1}{2}\cancel{m\omega^2}\frac{(qE)^2}{m^2\omega^4}$$

$$- \cancel{qE x'} - \frac{(qE)^2}{m\omega^2}$$

$$= \frac{1}{2}m\omega^2x'^2 + \frac{1}{2}\frac{(qE)^2}{m\omega^2} - \frac{(qE)^2}{m\omega^2}$$

$$= \frac{1}{2}m\omega^2x'^2 - \frac{1}{2}\frac{(qE)^2}{m\omega^2}$$

← note this term is independent of x (or x')

So the Schrodinger Equation becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx'^2} + \frac{1}{2}m\omega^2x'^2 = \left[E + \frac{1}{2}\frac{(qE)^2}{m\omega^2}\right]\psi$$

if we let $E' = E + \frac{1}{2}\frac{(qE)^2}{m\omega^2}$, we have recovered the original S.E., so

$$E_n' = (n+\tfrac{1}{2})\hbar\omega = E_n + \frac{1}{2}\frac{(qE)^2}{m\omega^2}$$

$$\therefore \quad E_n = (n+\tfrac{1}{2})\hbar\omega - \frac{1}{2}\frac{(qE)^2}{m\omega^2}$$

The Subtracted term is exactly what we got in part (a) by using P.T.

#2 (cont'd)

This problem gives us practice w/ 2nd order perturbation theory. It also shows that PT sometimes gives an exact result. Physical Significance of perturbation: The applied electric field has an interesting effect on the harmonic oscillator. First, it shifts all energy levels down by an amount $\frac{1}{2}\frac{(qE)^2}{m\omega^2}$. Secondly, it shifts the center of the H.O. over in x by an amount $\frac{qE}{m\omega^2}$ – this is the physical significance of the change of variable $x' = x - \frac{qE}{m\omega^2}$!