#1

Problem 11.2 Construct the analogs to Equation 11.12 for one-dimensional and two-dimensional scattering.

In 3-D: $\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}$ for large r.

The spherical wave has a $\frac{1}{r}$ dependence to ensure that $P(r,\theta) = |\psi|^2 \propto \frac{1}{r^2}$. This ensures that probability is conserved for large r.

- To conserve probability in 2-D we have a scattered cylindrical wave. To conserve probability flux in 2-D, $P \propto \frac{1}{r}$, so $\psi \propto \frac{1}{\sqrt{r}}$!

$$\boxed{\psi(r,\theta) \simeq A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{\sqrt{r}}\right\}}$$

- In 1-D, the scattered wave is also a plane wave, so there is no spatial dependence, as all probability flux stays on the x-axis: $P \propto 1$, so,

$$\boxed{\psi(x) \simeq A\left[e^{ikx} + f\left(\frac{x}{|x|}\right)e^{-ikx}\right]}$$

where the factor $\frac{x}{|x|}$ tells us we can only scatter in the + or − x-direction.

Relevance: This problem is intended to make us think about the reasonableness of our wavefunction approximation for scattering to large r in 3-D, by examining the nature of probability flux conservation in all dimensional cases.

#2

Problem 11.6 What are the partial wave phase shifts (δ_l) for hard-sphere scattering (Example 11.3)?

In Example 11.3, we find that the partial wave amplitude for hard-sphere scattering is

$$a_\ell = -i\,\frac{j_\ell(ka)}{k\,h_\ell^{(1)}(ka)} \qquad (1)$$

So we need to make use of the relation between the partial-wave amplitude a_ℓ & the phase shift δ_ℓ:

$$a_\ell = \frac{1}{k}\,e^{i\delta_\ell}\sin\delta_\ell \qquad (2)$$

to find the partial wave phase shifts.

Examining (1) & (2), it follows that

$$e^{i\delta_\ell}\sin\delta_\ell = i\,\frac{j_\ell(ka)}{h_\ell^{(1)}(ka)}$$

But (Eq. 11.19) $h_\ell^{(1)} = j_\ell(x) + i n_\ell(x)$, so

$$e^{i\delta_\ell}\sin\delta_\ell = i\,\frac{j_\ell(ka)}{j_\ell(ka) + i n_\ell(ka)} \left(= i\,\frac{j}{j+in} \text{ for short}\right)$$

Approach: write out both in terms of real & imaginary parts & equate.

$$\text{so,}\quad i\,\frac{j}{j+in} = i\,\frac{1}{1+i\frac{n}{j}} = i\,\frac{1-i\frac{n}{j}}{1+\left(\frac{n}{j}\right)^2} = \frac{\frac{n}{j}+i}{1+\left(\frac{n}{j}\right)^2}$$

$$\text{and}\quad e^{i\delta_\ell}\sin\delta_\ell = \cos\delta_\ell\sin\delta_\ell + i\sin^2\delta_\ell$$

So,

$$\cos\delta_\ell\sin\delta_\ell = \frac{n/j}{1+(n/j)^2} \quad \& \quad \sin^2\delta_\ell = \frac{1}{1+\left(\frac{n}{j}\right)^2}$$

#2 cont'd:

Now divide the second by the first:

$$\frac{\sin^2\delta_\ell}{\cos\delta_\ell \sin\delta_\ell} = \frac{\sin\delta_\ell}{\cos\delta_\ell} = \tan\delta_\ell = \frac{1}{\frac{\eta}{j}} = \frac{j_\ell(ka)}{\eta_\ell(ka)}$$

So,

$$\boxed{\delta_\ell = \tan^{-1}\left(\frac{j_\ell(ka)}{\eta_\ell(ka)}\right)}$$

Relevance: Exploration of partial wave & phase shift analysis for quantum mechanical scattering. Here we use an analytically approachable example (Hard sphere scattering) to relate phase shifts to partial wave amplitudes.

#3

Problem 11.10 Find the scattering amplitude, in the Born approximation, for soft-sphere scattering at arbitrary energy. Show that your formula reduces to Equation 11.82 in the low-energy limit.

The soft-sphere scattering potential is:

$$V(r) = \begin{cases} V_0 & \text{if } r \le a \\ 0 & \text{if } r > a \end{cases}$$

& since it is spherically symmetric, we can use the Born Approx.

for spherical symmetry (Eq. 11.88):

$$f(\theta) \cong -\frac{2m}{\hbar^2 \kappa} V_0 \int_0^a r \sin(\kappa r)\, dr$$

$$= -\frac{2mV_0}{\hbar^2 \kappa}\left[\frac{1}{\kappa^2}\sin(\kappa r) - \frac{r}{\kappa}\cos(\kappa r)\right]\Bigg|_0^a$$

$$= \boxed{-\frac{2mV_0}{\hbar^2 \kappa^3}\left[\sin(\kappa a) - \kappa a \cos(\kappa a)\right]}$$

For low energy scattering, $(\kappa a \ll 1)$,

$$\sin(\kappa a) \approx \kappa a - \frac{1}{3!}(\kappa a)^3$$

$$\cos(\kappa a) \approx 1 - \frac{1}{2}(\kappa a)^2$$

So,

$$f(\theta) \approx -\frac{2mV_0}{\hbar^2 \kappa^3}\left[\cancel{\kappa a} - \frac{1}{6}(\kappa a)^3 - \cancel{\kappa a} + \frac{1}{2}(\kappa a)^3\right]$$

$$= \boxed{-\frac{2}{3}\frac{mV_0 a^3}{\hbar^2}}$$, which is in agreement w/ Eq. 11.82. !

Relevance: To gain experience w/ the Born Approximation & its variations by exploring analytically solvable cases.

#4

*Problem 11.13 For the potential in Problem 11.4,

(a) calculate $f(\theta)$, $D(\theta)$, and σ, in the low-energy Born approximation;

(b) calculate $f(\theta)$ for arbitrary energies, in the Born approximation;

(c) show that your results are consistent with the answer to Problem 11.4, in the appropriate regime.

a) $V(r) = \alpha\delta(r-a)$

So from Eq. 11.80, $f = -\frac{m}{2\pi\hbar^2}\int V(\vec{r})\,d^3r$

$$= -\frac{m}{2\pi\hbar^2}\,\alpha\, 4\pi \int_0^\infty \delta(r-a)\, r^2\,dr$$

$$\boxed{f = -\frac{2m\alpha}{\hbar^2}a^2}$$

$$D = |f|^2 = \boxed{\left(\frac{2m\alpha}{\hbar^2}a^2\right)^2}$$

$$\sigma = 4\pi D = \boxed{\pi\left(\frac{4m\alpha}{\hbar^2}a^2\right)^2}$$

b) Eq 11.88 $\Rightarrow$ $f = -\frac{2m}{\hbar^2\kappa}\alpha\int_0^\infty r\,\delta(r-a)\sin(\kappa r)\,dr$

$$= \boxed{-\frac{2m\alpha}{\hbar^2\kappa}\, a\sin(\kappa a)}$$

c) Note first that b) reduces to a) in the low energy regime ($ka \ll 1 \Rightarrow \kappa a \ll 1$), as $\sin\kappa a \approx \kappa a$.

Now we need to show that the result of a) is consistent w/ problem 11.4 when in the regime of the Born Approximation. This regime is that in which the potential is weak. Inspecting the result of 11.4:

$$\sigma = \frac{4\pi a^2\beta^2}{(1+\beta)^2} \quad ; \quad \beta = \frac{2ma\alpha}{\hbar^2}$$

#4 cont'd:

we see that the size of the potential (α) is included in the term β. So the Born Appx amounts to $\beta \ll 1$

Then, $\sigma \approx 4\pi a^2\beta^2 = \dfrac{16\pi m^2 a^4 \alpha^2}{\hbar^4} = \pi\left(\dfrac{4m\alpha}{\hbar^2}a^2\right)^2$,

which is in agreement w/ (a).

<u>Relevance</u>:

To gain experience w/ the Born Approximation & its variation by exploring analytically solvable cases.